Evolution of a passive scalar spectrum in the flow of random waves

Gregory Falkovich and Doron Shlomo
Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel
(Received 30 March 2005; published 30 June 2005)

We consider a passive pollutant advected by the flow due to linear random waves with finite attenuation. We derive the equation that governs the evolution of the pair correlation function of pollutant concentration and show that it coincides with the equation for the case of a short-correlated velocity. Due to a finite wave attenuation, nontrivial evolution (particularly, the growth of inhomogeneities) appears already in the second order in wave amplitudes. We show that random potential waves lead to the growth of concentration inhomogeneities. We identify two stationary solutions for the spectral density of concentration, equipartition, and flux state. Which one is established depends on the relation between mean square velocity gradients due to potential and solenoidal parts of the flow, respectively. We also analyze transient regimes and show how periodic component in the concentration distribution appears and disappears.

DOI: 10.1103/PhysRevE.71.067304 PACS number(s): 47.27.Qb, 05.40.−a

In this work we consider the advection of a passive scalar by a flow which is a superposition of small-amplitude random waves with finite attenuation rates. An example of such a scalar field is the concentration of pollutant on a fluid surface and by passive we mean that the velocity field does not depend on the concentration.

In a random flow, one expects some mixing [1–6], which makes the concentration more uniform. On the other hand, when the flow is compressible (which is the case for waves on the water) it creates fluctuations of density [6–8]. The main question: is there a statistical steady state of concentration fluctuation where production of fluctuations is balanced by mixing?

Since the mean concentration does not change, the simplest statistical object to study is the second moment. Here we derive the equation that describes the evolution of the pair correlation function of the passive concentration in a random flow of small-amplitude waves with finite attenuation. We show that when the velocity in the wave has a potential component (i.e., the flow is compressible) the second moment of the concentration grow exponentially that is there are no steady states with a finite mean squared density (such grows is stopped by molecular diffusion [7] which is beyond the present consideration). We then show that one can find steady states which correspond to an infinite mean squared density. A simplest such state is an equipartition in **k** space which corresponds to concentration fluctuations delta correlated in **r** space. We show that there exists another steady-state solution, apart from equipartition. At the wave numbers far exceeding those of the waves, the spectral density of the scalar in that second solution depends on the wave number by a power law. We show that this power law is determined by the ratio of mean square velocity gradients, respectively, in the solenoidal and potential components of the flow (which precisely expresses the relative strength of mixing versus fluctuation production). We argue that which steady solution is realized depends on that ratio.

Apart from an asymptotic-in-time approach to a steady state, it is of both fundamental and practical interest to study transient processes. It is particularly interesting when waves have a sharp spectral peak. Does that impose some periodicity on the distribution of passive density? We shall study the evolution of the scalar spectral density assuming the initial state to have wave numbers much lower than those of the waves. In this case, the large-scale fluctuations of the passive scalar decay due to simple diffusion. We shall show, both analytically and numerically, that a series of peaks appear in the spectral density of passive scalar at wave numbers which are integer times the peak wave number of the waves. We shall see in numerical simulations in the simplest one-dimensional (1D) case how those peaks appear and eventually disappear and how equipartition is settled.

Consider a concentration $n(\mathbf{r},t)$ that satisfies the continuity equation

$$\frac{\partial n}{\partial t} = -\operatorname{div} n\mathbf{v}. \tag{1}$$

We assume the velocity statistics to be Gaussian with zero mean and the variance (in Fourier representation)

$$\langle v^m_{\mathbf{k}\omega} v^m_{\mathbf{k}'\omega'}\rangle = E_{mn}(\mathbf{k},\omega)\,\delta(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega'). \tag{2}$$

We assume that $E_{mn}(\mathbf{k},\omega)$ has sharp peaks at $\omega=\pm\omega_k$ which takes place, in particular, for small-amplitude waves with the frequency ω_k and a weak attenuation rate ($\gamma_k \ll \omega_k$) when

$$E_{mn}(\mathbf{k},\omega) = \gamma_k E_{mn}(\mathbf{k})\{[(\omega-\omega_k)^2+\gamma_k^2]^{-1}+[(\omega+\omega_k)^2 + \gamma_k^2]^{-1}\}. \tag{3}$$

As long as $E_{mm}(\mathbf{k}) = \int E_{mm}(\mathbf{k},\omega)d\omega \ll \omega_k^2/k^2$ (which means that fluid velocity is far lass than the phase velocity of waves), one can apply the perturbation theory to (1) and derive the equation for the spectral density of the concentration $\langle n_{\mathbf{k}} n_{\mathbf{k}'}\rangle = H_{\mathbf{k}}\delta(\mathbf{k}+\mathbf{k}')$ [8,9]:

$$\frac{\partial H_{\mathbf{k}}}{\partial t} = k^n k^m \int (H_{\mathbf{k}+\mathbf{q}} - H_{\mathbf{k}-\mathbf{q}} - 2H_{\mathbf{k}})D_{mn}(\mathbf{q})d\mathbf{q}, \tag{4}$$

where $D_{mn}(\mathbf{q}) = E_{mn}(\mathbf{q},0) = E_{mn}(\mathbf{q})\gamma_k/\pi\omega_k^2$.

For the Kraichnan model of delta-correlated velocity, the simultaneous pair correlation function of the passive density $H(\mathbf{r}) = \langle n(\mathbf{r})n(0)\rangle$ satisfies the equation [6]

$$\frac{\partial H(\mathbf{r},t)}{\partial t} = \nabla_m \nabla_n d_{mn}(\mathbf{r}) H(\mathbf{r},t), \tag{5}$$

where $d_{mn}(\mathbf{r}) = D_{mn}(\mathbf{r}) - D_{mn}(0)$ and

$$D_{nm}(\mathbf{r}) = \int_0^\infty dt \langle v_n(\mathbf{R},t) v_m(0,0)\rangle. \tag{6}$$

The correlation function in (6) must be taken in the Lagrangian frame $\dot{\mathbf{R}} = \mathbf{v}(\mathbf{R})$, $\mathbf{R}(0) = \mathbf{r}$ [10], which is close to the Eulerian correlation function in a short-correlated case. Note that after the Fourier transform Eq. (4) coincides with (5) assuming $D_{mn}(\mathbf{r}) = \int D_{mn}(\mathbf{k}) \exp[i(\mathbf{k}\cdot\mathbf{r})] d\mathbf{k}$.

Let us briefly present the general properties of (4). First, it conserves $H(k=0)$, which is the total mass. The kernel $k_n k_m D_{nm} = k^2[S(q)\sin^2\theta_{kq} + P(q)\cos^2\theta_{kq}]$ is positive so that (4) satisfies the maximum principle, that is, the solution cannot break out of the limits: if $m \leqslant H(\mathbf{k},t_0) \leqslant M$ for all $\mathbf{k}$, then for any $t > t_0$, $m \leqslant H(\mathbf{k},t) \leqslant M$ for all $\mathbf{k}$. Large-scale fluctuations of the density simply diffuse if for $k \ll q$ one has $H(k) \gg H(q)$: $\partial H(k,t)/\partial t = -Dk^2 H(k,t)$ where

$$D = \int [S(q)\sin^2\theta_{kq} + P(q)\cos^2\theta_{kq}] dbfq. \tag{7}$$

The most interesting property of (4) is the growth of density inhomogeneities due to the potential part of the flow

$$\frac{dN}{dt} = 2\lambda N, \quad \lambda = \int q_n q_m D_{nm}(\mathbf{q}) d\mathbf{q} = \int q^2 P(q) d\mathbf{q}. \tag{8}$$

Note that the growth rate of the second moment is exactly minus twice the entropy production rate determined by the pair-correlation function of $\omega = \mathrm{div}\,\mathbf{v}$ [10]: $\lambda = \int \langle \omega(\mathbf{R},t)\omega(0,0)\rangle dt \approx \int \langle \omega(0,t)\omega(0,0)\rangle dt$. One can roughly estimate the growth rate of density fluctuations as follows:

$$\lambda \simeq \langle\langle (\mathrm{div}\,\mathbf{v})^2\rangle \gamma_q \omega_q^{-2} \simeq \omega_k (\gamma_k/\omega_k)(k^2 E(k)/\omega_k)^2. \tag{9}$$

To give it a numerical value, take short gravity waves with periods in seconds, $\gamma_k/\omega_k \sim 10^{-3}$ and $kv/\omega_k \simeq 10^{-1}$ then $1/\lambda \simeq 10^5$ s, i.e., a couple of hours. One can also compare (8) with the estimate $\lambda \sim \langle (\mathrm{div}\,\mathbf{v})^2\rangle \langle (\mathrm{curl}\, v)^2\rangle \omega_q^{-3}$ obtained in Ref. [8], which gives zero for purely potential waves.

Let us describe now the evolution of the passive scalar spectral density. There is an evident steady solution of (4) which is equipartition: $H_k = \mathrm{const}$. It corresponds to $H(\mathbf{r}) \propto \delta(\mathbf{r})$. Are there other steady solutions?

We consider an isotropic case and presume that both solenoidal and potential components are present in the flow

$$D_{nm}(\mathbf{q}) = S(q)\left(\delta_{nm} - \frac{q_n q_m}{q^2}\right) + P(q)\frac{q_n q_m}{q^2}. \tag{10}$$

The Fourier transform is convenient to represent via two other functions, $u(r)$ and $c(r)$,

$$d_{\alpha\beta}(r) = \left[\frac{(r^3 u)'}{r^2} - c\right] r^2 \delta_{\alpha\beta} - \left[\frac{(r^2 u)'}{r} - c\right] r_\alpha r_\beta,$$

so that we can write (5) in the spherical coordinates

$$\partial_t H = r^{-1}\hat{\mathcal{L}}(rH) = r^{-1}\partial_r r^3 u H_{\mathrm{st}} \partial_r (H/H_{\mathrm{st}}), \tag{11}$$

$$H_{\mathrm{st}}(r) = \exp\left[\int_r^\infty \frac{c(r')dr'}{r' u(r')}\right]. \tag{12}$$

One can show that the operator $\hat{\mathcal{L}}$ has no positive eigenvalues [7] and that the solution of (11) evolves towards the steady state H_{st}, as long as the solution grows at $r \to 0$ not faster than r^{-2} so that $H(k=0) = \int H(r) d\mathbf{r}$ is finite. That requires $c(0)/u(0) < 2$. Note that $c(0) = \lambda/24$. Denote also $\tilde{\lambda} = \int q^2 S(q) d\mathbf{q}$. Then $c(0)/u(0) = 8\lambda/(3\lambda + \tilde{\lambda}) < 2$ means $\lambda < \tilde{\lambda}$. One can say that λ characterizes production of fluctuations while $\tilde{\lambda}$ characterizes mixing. Note in passing the Lyapunov exponents of the Lagrangian flow in this case: $\lambda_{1,2} = -\lambda/2 \pm (\tilde{\lambda} + 3\lambda)/16$.

Therefore, we see that if the mean squared gradient of the solenoidal part exceeds that of the potential part of the flow, the steady-state solution is not an equipartition. Indeed,

$$H_{\mathrm{st}}(k) \propto k^{2(\lambda - \tilde{\lambda})/(\tilde{\lambda} + 3\lambda)} \text{ at } k \to \infty. \tag{13}$$

In a purely solenoidal case, $H_{\mathrm{st}}(k) \propto k^{-2}$ is called the Batchelor spectrum; it carries a constant flux of n^2 towards large k and presumes some source of fluctuations at large scales [11]. And generally at $\lambda < \tilde{\lambda}$, (13) decays at $k \to \infty$, that is, corresponds to some flux from small to large k while equipartition evidently carries zero flux. Note that when $\lambda \neq 0$ one does not need an external source at large scales (as in purely solenoidal case) since the potential part of the flow creates concentration fluctuations.

Let us now consider (4) at wave numbers exceeding those in the spectrum of waves in more detail. Assuming that the spectral energy of waves $E(k)$ falls off faster than any power (say, exponentially) we can use the differential approxima-

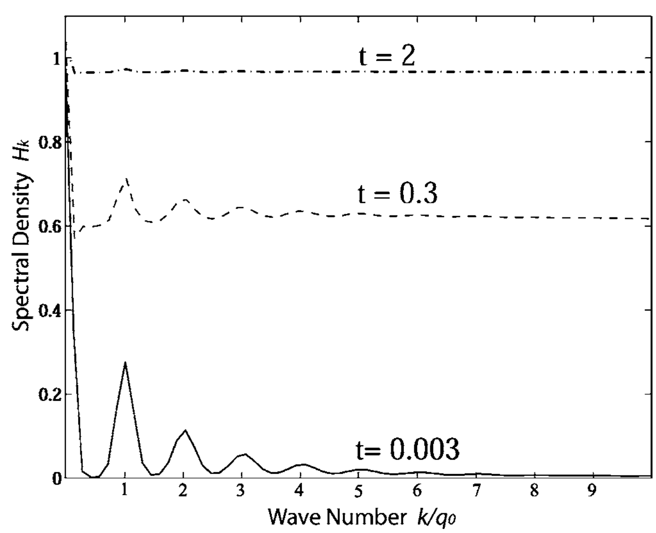

FIG. 1. Evolution of spectral density of passive scalar in 1D. Waves have a spectral peak at $k=1$.

tion in q/k (also called the Batchelor regime, which considered small-scale fluctuations of passive scalar in the large-scale velocity field [11]):

$$8\frac{\partial H(k,t)}{\partial t} = (\tilde{\lambda} + 3\lambda)k^2\frac{\partial^2 H(k,t)}{\partial k^2} + (3\tilde{\lambda} + \lambda)k\frac{\partial H(k,t)}{\partial k}. \tag{14}$$

In logarithmic coordinates in a moving reference frame, $\zeta \equiv \ln(k) + (\tilde{\lambda} - \lambda)t/4$, one obtains a simple diffusion equation:

$$\frac{\partial H(\zeta,t)}{\partial t} = \frac{(\tilde{\lambda} + 3\lambda)}{8}\frac{\partial^2 H(\zeta,t)}{\partial \zeta^2}. \tag{15}$$

Note that λ and $\tilde{\lambda}$ are non-negative quantities so that small-scale harmonics of passive density undergo diffusion in k space (in logarithmic coordinates).

An appropriate solution is $H(k,t) \propto \mathrm{erf}\{-[4\ln k + (\tilde{\lambda} - \lambda)t]^2/8(\tilde{\lambda} + 3\lambda)t\}$ when the front velocity, $\lambda - \tilde{\lambda}$, is positive. It is in this case that the equipartition is established while H_{st} does not have a physical meaning since it corresponds to an infinite integral. This is realized in particular in the one-dimensional case analyzed in Ref. [12]. On the contrary, when $\lambda < \tilde{\lambda}$, the solution H_{st} decays faster than equipartition at $k \to \infty$ and it is established.

Of course, there may be many interesting transient processes on the way to the steady state [2]. A particularly interesting case is when wave spectrum $D(\mathbf{q})$ has a peak around some $\mathbf{q}_0$ while initially only large-scale fluctuations of the scalar are present, i.e., $H(\mathbf{k},0)$ is nonzero only for $k \ll q$. In this case, it is clear from (4) that initially $H(\mathbf{k},t)$ starts to grow at $\mathbf{k} \approx \mathbf{q}_0$ with $\partial H(\mathbf{k})/\partial t \propto H(0)D(\mathbf{q})$, then the peak around $\mathbf{k} = 2\mathbf{q}_0$ will grow, etc. Such an evolution for the one-dimensional (purely potential) case is shown in Fig. 1 with equipartition eventually covering the whole spectrum.

[1] K. Herterich and K. Hasselmann, J. Phys. Oceanogr. **12**, 704 (1982).

[2] P. B. Weichman and R. E. Glazman, Phys. Rev. Lett. **83**, 5011 (1999); J. Fluid Mech. **420**, 147 (2000); **453**, 263 (2002).

[3] A. M. Balk and R. M. McLaughlin, Phys. Lett. A **256**, 299 (1999).

[4] R. Ramshankar, D. Berlin, and J. P. Gollub, Phys. Fluids A **2**, 1955 (1990).

[5] E. Schröder, J. S. Andersen, M. T. Levinsen, P. Alstrøm, and W. I. Goldburg, Phys. Rev. Lett. **76**, 4717 (1996).

[6] G. Falkovich, K. Gawedzki, and M. Vergassola, Rev. Mod. Phys. **73**, 913 (2001).

[7] E. Balkovsky, G. Falkovich, and A. Fouxon, e-print chao-dyn/9912027; Phys. Rev. Lett. **86**, 2790 (2001).

[8] A. Balk, G. Falkovich, and M. Stepanov, Phys. Rev. Lett. **92**, 244504 (2004).

[9] A. Balk, J. Fluid Mech. **467**, 163 (2002).

[10] G. Falkovich and A. Fouxon, New J. Phys. **6**, 50 (2004); e-print nlin.CD/0312033 (2003).

[11] G. K. Batchelor, J. Fluid Mech. **5**, 113 (1959).

[12] M. Wilkinson and B. Mehlig, Phys. Rev. E **68**, 040101(R) (2003).